Krivoliniski integral prve vrste (po luku)

Ako je c kriva data u ravni opisana jednačinom $y=\eta(x)$ gdje je $a \le x \le b$ tada

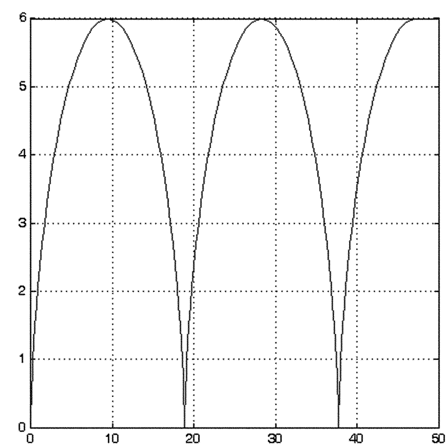
$$\int_c f(x,y)\,ds = \int_a^b f(x,\eta(x))\underbrace{\sqrt{1+(\eta'(x))^2}\,dx}_{ds}$$

Ako je c kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1 \le t \le t_2$ tada

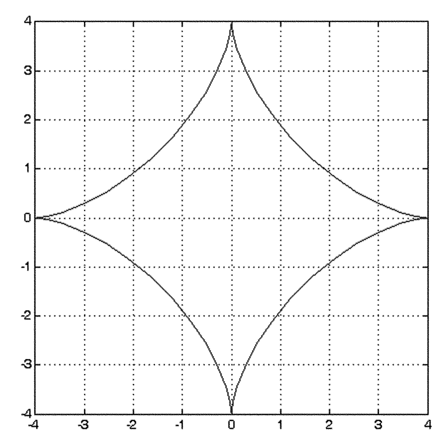
$$\boxed{\mu'(t)=\frac{\partial \mu}{\partial t}}$$

$$\int_c f(x,y)\,ds = \int_{t_1}^{t_2} f(\mu(t),\eta(t))\underbrace{\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt}_{ds}$$

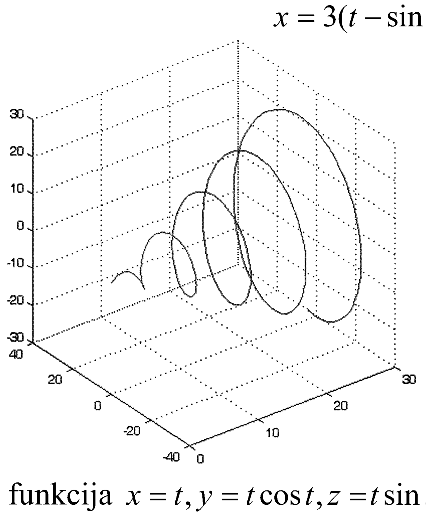
Krivoliniski integrali prve vrste f-ja triju promenjivih $f(x,y,z)$ uzeti po prostornoj krivoj se računaju analogno. Krivoliniski integral prve vrste NE OVISI O SMJERU PUTA INTEGRACIJE.



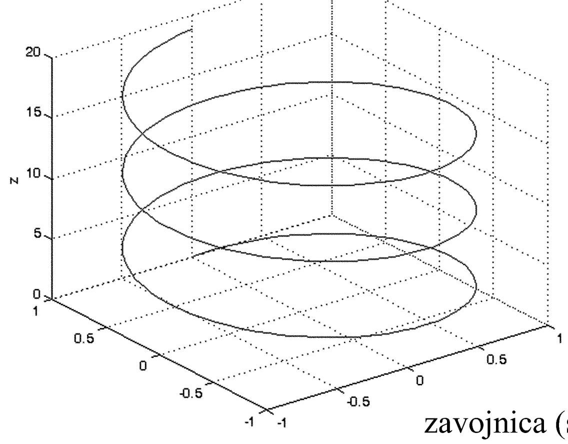
cikloida
$x=3(t-\sin t), y=3(1-\cos t)$, $0 \le t \le 5\pi$.



funkcija $x=4\cos^3 t, y=4\sin^3 t$, $0 \le t \le 2\pi$.



funkcija $x=t, y=t\cos t, z=t\sin t$, $0 \le t \le 30$.



zavojnica (spirala)
$x=\sin t, y=\cos t, z=t$, $0 \le t \le 6\pi$.

(#) Izračunati krivolinijski integral $I=\int_L (4\sqrt[3]{x}-3\sqrt{y})\,dl$

između tački $E(-1;0)$ i $F(0;1)$

a) po pravoj EF;

b) po liniji astroide $x=\cos^3 t$, $y=\sin^3 t$.

Rj.

$$I=\int_L (4\sqrt[3]{x}-3\sqrt{y})\,dl$$

Ovo je krivolinijski integral prve vrste. Prisjetimo se

Ako je L kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$ tada

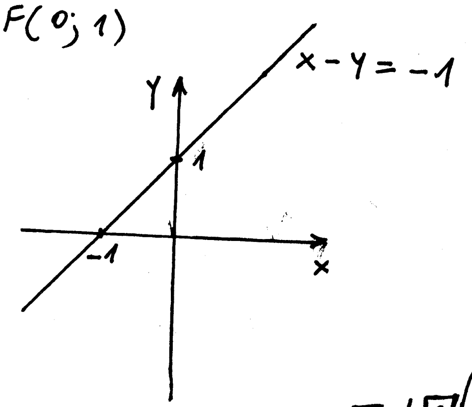
$$\int_L f(x,y)\,dl=\int_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

Ako je L opisana parametarskim jednačinama $\begin{cases} x=\mu(t) \\ y=\theta(t) \end{cases}$ gdje $t_1\le t\le t_2$

$$\int_L f(x,y)\,dl=\int_{t_1}^{t_2} f(\mu(t),\theta(t))\sqrt{(\mu'(t))^2+(\theta'(t))^2}\,dt$$

a) $E(-1;0)$

$F(0;1)$



$-y=-x-1$, $x\in[-1,0]$ tj. $y=x+1$

$y'=1 \Rightarrow dl=\sqrt{1+1^2}\,dx=\sqrt{2}\,dx$

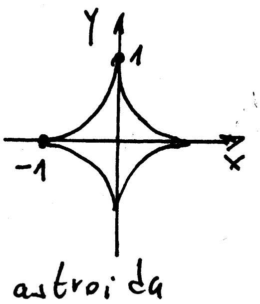
$$I=\int_L (4\sqrt[3]{x}-3\sqrt{y})\,dl=\int_{-1}^{0}(4x^{\frac{1}{3}}-3(x+1)^{\frac{1}{2}}\sqrt{2}\,dx$$

$$=4\sqrt{2}\int_{-1}^{0}x^{\frac{1}{3}}dx-3\sqrt{2}\int_{-1}^{0}(x+1)\,dx=$$

$$=4\sqrt{2}\cdot\frac{3}{4}x^{\frac{4}{3}}\Big|_{-1}^{0}-3\sqrt{2}\int_{-1}^{0}(x+1)^{\frac{1}{2}}d(x+1)=3\sqrt{2}(0-1)-3\sqrt{2}\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_{-1}^{0}=-5\sqrt{2}$$

traženo rješenje

b)



astroida

$x=\cos^3 t$, $x'=-3\cos^2 t\sin t$

$y=\sin^3 t$, $y'=3\sin^2 t\cos t$

$$dl=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt$$

$$\sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\sqrt{\cos^2 t \sin^2(t)\underbrace{(\cos^2 t + \sin^2 t)}_{=1}} =$$

$$= 3\,|\sin t \cos t|$$

U našem slučaju t uzima vrijednost od $\frac{\pi}{2}$ do π, pa je

$$dl = -3\sin t \cos t\, dt$$

$$I = \int_L \left(4\sqrt[3]{x} - 3\sqrt{y}\right) dl = \int_{\frac{\pi}{2}}^{\pi} \left(4\sqrt[3]{\cos^3 t} - 3\sqrt{\sin^3 t}\right)(-3\sin t \cos t)\, dt$$

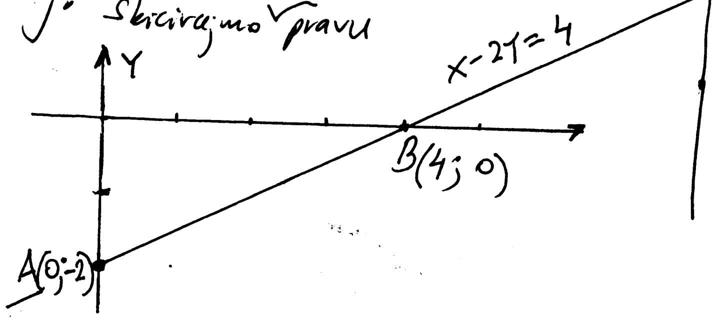
$$= -12\int_{\frac{\pi}{2}}^{\pi} \cos^2 t \sin t\, dt + 9\int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{5}{2}} t \cos t\, dt =$$

$$= +12\int_{\frac{\pi}{2}}^{\pi} \cos^2 t\, d\cos t + 9\int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{5}{2}} t\, d\sin t = 12\cdot \frac{\cos^3 t}{3}\Big|_{\frac{\pi}{2}}^{\pi} + 9\cdot \frac{\sin^{\frac{7}{2}} t}{\frac{7}{2}}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 4\left((-1)^3 - 0\right) + \frac{18}{7}\left(0 - 1^{\frac{7}{2}}\right) = -4 - \frac{18}{7} = -\frac{46}{7}$$ traženo rješenje

Izračunati krivoliniski integral $\int\limits_{AB} \frac{dl}{\sqrt{x^2+y^2}}$ po odsječku prave $x-2y=4$ od tačke $A(0;-2)$ do tačke $B(4;0)$.

Rj. Skicirajmo datu pravu



Prisjetimo se kako se računa krivoliniski integral prvog tipa, ako je kriva integracije c u ravni opisana formulom $y=\eta(x)$, $a \le x \le b$

$$\int\limits_c f(x,y)\,dl = \int\limits_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

I način:

$x-2y=4$

$2y=x-4$

$y=\frac{1}{2}x-2$

$y'=\frac{1}{2}$

$$\int\limits_{AB}\frac{dl}{\sqrt{x^2+y^2}} = \int\limits_0^4 \frac{\sqrt{1+\frac{1}{4}}}{\sqrt{x^2+(\frac{1}{2}x-2)^2}}\,dx = \frac{\sqrt{5}}{2}\int\limits_0^4 \frac{dx}{\sqrt{\frac{5x^2}{4}-2x+4}}$$

$$= \frac{\sqrt{5}}{2}\cdot\frac{1}{\sqrt{\frac{5}{4}}}\int\limits_0^4\frac{dx}{\sqrt{x^2-\frac{8}{5}x+\frac{16}{5}}} = \left| \begin{array}{l} x^2-\frac{8}{5}x+\frac{16}{5} = \\ = x^2-2\cdot x\cdot\frac{4}{5}+\frac{16}{25}-\frac{16}{25}+\frac{16\cdot 5}{5\cdot 5} \\ = (x-\frac{4}{5})^2+\frac{64}{25} \end{array} \right| ^{80}$$

$$= \int\limits_0^4 \frac{d(x-\frac{4}{5})}{\sqrt{(x-\frac{4}{5})^2+\frac{64}{25}}} = \ln\left|x-\frac{4}{5}+\sqrt{(x-\frac{4}{5})^2+\frac{64}{25}}\right|\Bigg|_0^4 = \ln\left(\frac{16}{5}+\sqrt{\frac{16(16+4)}{25}}\right) -$$

$$- \ln\left(-\frac{4}{5}+\sqrt{\frac{\overset{16(1+4)}{16+64}}{25}}\right) = \ln\left(\frac{16}{5}+\frac{8\sqrt{5}}{5}\right) - \ln\left(-\frac{4}{5}+\frac{4\sqrt{5}}{5}\right)$$

$$= \ln\frac{\frac{16+8\sqrt{5}}{\cancel{5}}}{\frac{-4+4\sqrt{5}}{\cancel{5}}} = \ln\frac{4+2\sqrt{5}}{\sqrt{5}-1}\,\frac{/\cdot(\sqrt{5}+1)}{/\cdot(\sqrt{5}+1)} = \ln\frac{4+6\sqrt{5}+10}{5-1} = \ln\frac{7+3\sqrt{5}}{2}$$ tražena vrijednost

II način

$x-2y=4$

$x=2y+4$

$\frac{\partial x}{\partial y}=2$

$$\int\limits_{AB}\frac{dl}{\sqrt{x^2+y^2}} = \int\limits_{-2}^0\frac{\sqrt{1+4}}{\sqrt{(2y+4)^2+y^2}}\,dy = \sqrt{5}\int\limits_{-2}^0\frac{dy}{\sqrt{5y^2+16y+16}} = \ldots$$

ZAVRŠITI ZA VJEŽBU

(#) Izračunati krivoliniski integral $\int_L (x-y)\,ds$ po kružnoj liniji $x^2+y^2=ax$.

Rj.

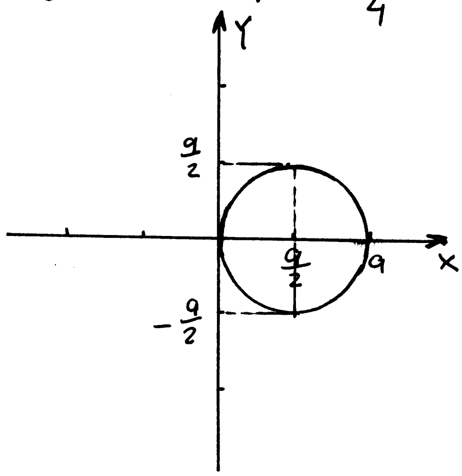
$$x^2+y^2=ax$$

$$x^2-ax+y^2=0$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$

krug sa centrom u $C(\frac{a}{2},0)$ poluprečnika $r=\frac{a}{2}$



Kako se računa krivoliniski integral $\int_L f(x,y)\,ds$?

Ako je kriva L data u obliku f-je $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_L f(x,y)\,ds=\int_a^b f(x,\eta(x))\sqrt{1+(y')^2}\,dx.$$

Ako je kriva L data u parametarskom obliku

$$\begin{cases} x=\mu(t) \\ y=\theta(t) \\ t_1\le t\le t_2 \end{cases}$$

tada $\int_L f(x,y)\,ds=\int_{t_1}^{t_2} f(\mu(t),\theta(t))\sqrt{\mu'(t)^2+\theta'(t)^2}\,dt$

Prisjetimo se polarnih koordinata $x=r\cos\varphi$, $y=r\sin\varphi$.

Ako pomjerimo centar u x-osi za $\frac{a}{2}$ i fiksiramo r na $\frac{a}{2}$ imamo da je

$$L:\begin{cases} x=\frac{a}{2}+\frac{a}{2}\cos\varphi \\ y=\frac{a}{2}\sin\varphi \\ 0\le\varphi\le 2\pi \end{cases}$$

$$x'=-\frac{a}{2}\sin\varphi \qquad y'=\frac{a}{2}\cos\varphi$$

$$(x')^2+(y')^2=\frac{a^2}{4}\sin^2\varphi+\frac{a^2}{4}\cos^2\varphi=\frac{a^2}{4}$$

$$\sqrt{x'^2+y'^2}=\frac{a}{2}$$

$$\int_L (x-y)\,ds=\int_0^{2\pi}\left(\frac{a}{2}+\frac{a}{2}\cos\varphi-\frac{a}{2}\sin\varphi\right)\cdot\frac{a}{2}\,d\varphi=\int_0^{2\pi}\left(\frac{a^2}{4}+\frac{a^2}{4}\cos\varphi-\frac{a^2}{4}\sin\varphi\right)d\varphi=$$

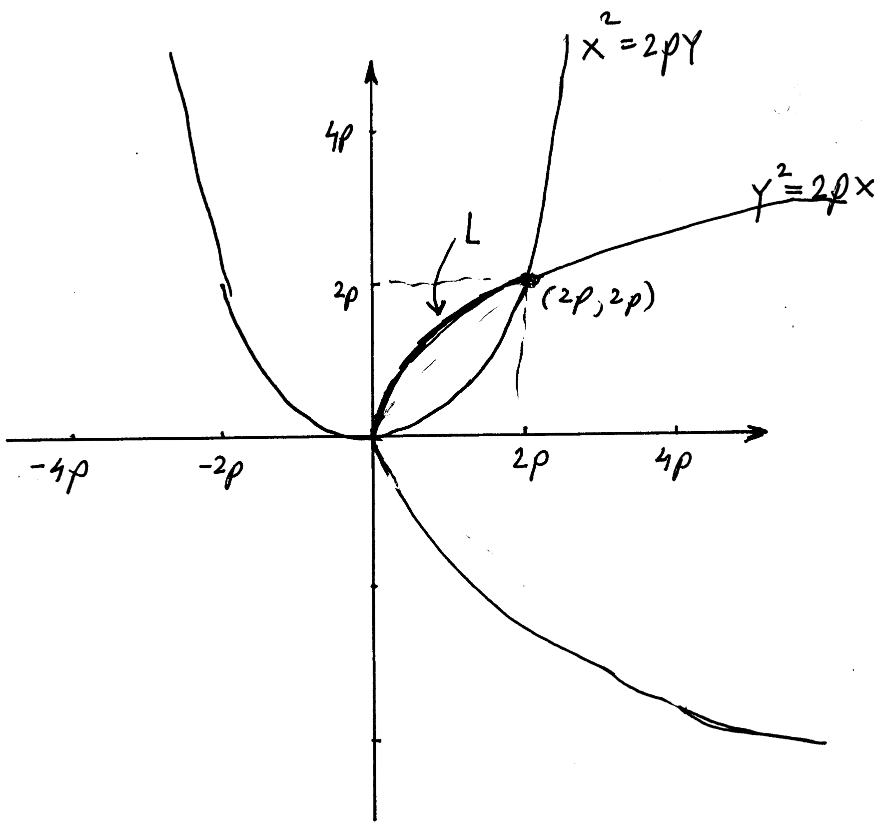
$$=\frac{a^2}{4}\varphi\Big|_0^{2\pi}+\frac{a^2}{4}\sin\varphi\Big|_0^{2\pi}+\frac{a^2}{4}\cos\varphi\Big|_0^{2\pi}=\frac{a^2}{4}\cdot 2\pi=\frac{a^2\pi}{2}$$

traženo rješenje

(#) Izračunati krivoliniski integral $\int_L y\,ds$ pri čemu je L luk parabole $y^2=2px$, koji leži unutar parabole $x^2=2py$.

Rj.

Skicirajmo dvije date parabole:



Prisjetimo se kako se računa krivolinijski integral $\int_L f(x,y)\,ds$.

Ako je L kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$ se računa

$$\int_L f(x,y)\,ds = \int_a^b f(x,\eta(x))\underbrace{\sqrt{1+(\eta'(x))^2}\,dx}_{ds}$$

U našem slučaju $y=\sqrt{2px}$ gdje je $0\le x\le 2p$

$$y=\sqrt{2px}=\sqrt{2p}\cdot\sqrt{x} \qquad (y')^2=\frac{2p}{4x}=\frac{p}{2x}$$

$$y'=\sqrt{2p}\cdot\frac{1}{2\sqrt{x}}=\frac{p}{\sqrt{2px}} \qquad 1+(y')^2=1+\frac{p}{2x}=\frac{2x+p}{2x}$$

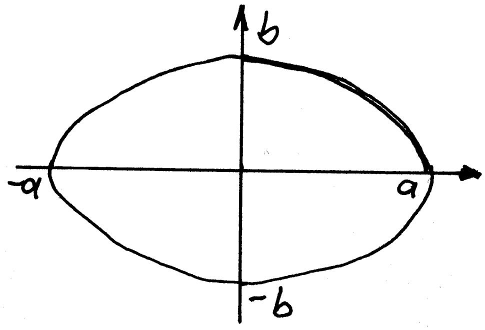
$$\int_L y\,ds=\int_0^{2p}\sqrt{2p}\cdot\sqrt{x}\cdot\frac{\sqrt{2x+p}}{\sqrt{2x}}\,dx=\sqrt{2}\cdot\sqrt{p}\cdot\frac{1}{\sqrt{2}}\int_0^{2p}\sqrt{x}\,\frac{\sqrt{2x+p}}{\sqrt{x}}\,dx=\left|\begin{matrix}d(2x+p)=\\=2dx\end{matrix}\right|$$

$$=\sqrt{p}\int_0^{2p}\sqrt{2x+p}\cdot\frac{1}{2}\,d(2x+p)=\frac{\sqrt{p}}{2}\cdot\frac{2}{3}\,(2x+p)^{\frac{3}{2}}\Big|_0^{2p}=\frac{p^{\frac{1}{2}}}{3}\left((5p)^{\frac{3}{2}}-p^{\frac{3}{2}}\right)$$

$$=\frac{1}{3}\cdot p^{\frac{1}{2}}\cdot p^{\frac{3}{2}}\left(\sqrt{5^3}-1\right)=\frac{p^2}{3}\left(5\sqrt{5}-1\right)$$ traženo rješenje

#) Izračunati $\int\limits_c xy\,ds$ gdje je c četvrtina elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ koja leži u prvom kvadrantu.

Rj. I način:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2-\frac{b^2}{a^2}x^2$$

$$y=\pm\sqrt{b^2\left(1-\frac{1}{a^2}x^2\right)}$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$c:\begin{cases} y=\frac{b}{a}\sqrt{a^2-x^2} \\ 0\le x\le a\end{cases}$$

$$y'=\frac{b}{a}\cdot\frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}$$

$$\int\limits_c xy\,ds=\int_0^a x\frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\left(\frac{b}{a}\cdot\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dx$$

$$=\dots$$

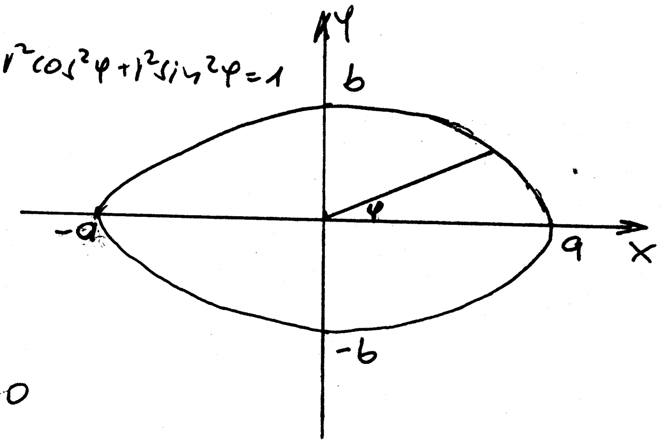
II način

Uvodimo poopštene polarne koordinate

$$x=ar\cos\varphi \qquad x^2=a^2r^2\cos^2\varphi \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2\cos^2\varphi+r^2\sin^2\varphi=1$$
$$y=br\sin\varphi \qquad y^2=b^2r^2\sin^2\varphi$$

$$\Downarrow$$

$$\left.\begin{array}{l}\text{za } \varphi=0 \text{ imamo } x=a,\ y=0\\ \text{za } \varphi=\frac{\pi}{2} \text{ imamo } x=0,\ y=b\end{array}\right\}\Rightarrow r=1$$



Sad elipsu možemo napisati u parametarskom obliku tj. imamo

$$c:\begin{cases} x=a\cos\varphi \\ y=b\sin\varphi \\ 0\le\varphi\le\frac{\pi}{2}\end{cases} \qquad \frac{\partial x}{\partial\varphi}=-a\sin\varphi \qquad \frac{\partial y}{\partial\varphi}=b\cos\varphi$$

$$\boxed{\int\limits_c f(x,y)\,ds=\int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\psi}{\partial t}\right)^2}\,dt \quad \text{gdje je} \quad \begin{array}{l} x=\varphi(t)\\ y=\psi(t)\\ \alpha\le t\le\beta\end{array}}$$

$$\int\limits_c xy\,ds=\int_0^{\pi/2}(a\cos\varphi)(b\sin\varphi)\sqrt{a^2\underbrace{\sin^2\varphi}_{1-\cos^2\varphi}+b^2\cos^2\varphi}\,d\varphi=ab\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\sqrt{a^2+(b^2-a^2)\cos^2\varphi}\,d\varphi$$

$$=\left|\begin{array}{ll} a^2+(b^2-a^2)\cos^2\varphi=u & \varphi=0\Rightarrow u=b^2\\ (b^2-a^2)2\cos\varphi(-\sin\varphi)\,d\varphi=du & \varphi=\frac{\pi}{2}\Rightarrow u=a^2\end{array}\right|=ab\cdot\frac{-1}{2(b^2-a^2)}\int_{b^2}^{a^2}\sqrt{u}\,du$$

$$=\frac{-ab}{2(b^2-a^2)}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{b^2}^{a^2}=\frac{-ab}{(b-a)(b+a)}\cdot\frac{1}{3}\cdot\underbrace{(a^3-b^3)}_{(a-b)(a^2+ab+b^2)}=\frac{ab}{3(a+b)}(a^2+ab+b^2)$$

(#) Izračunati krivolinijski integral $I = \int_c z\sqrt{x^2+y^2+2z^2}\,dS$

ako je c kriva $x = \frac{r\sqrt{2}}{2}\cos t$,

$y = \frac{r\sqrt{2}}{2}\cos t$, $z = r\sin t$, $t \in [0, \pi]$.

Rj. Ako je c kriva opisana parametarskim jednačinama

$$c: \begin{cases} x = \mu(t) \\ y = \eta(t) \\ z = \psi(t) \end{cases}, \quad t_1 \le t \le t_2 \quad \text{tada}$$

$$\int_c f(x,y,z)\,dS = \int_{t_1}^{t_2} f(\mu(t),\eta(t),\psi(t)) \underbrace{\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\psi'(t))^2}\,dt}_{dS}$$

$$x^2+y^2+2z^2 = \frac{2r^2}{4}\cos^2 t + \frac{2r^2}{4}\cos^2 t + 2r^2\sin^2 t = r^2\cos^2 t + 2r^2\sin^2 t$$

$$x'_t = -\frac{\sqrt{2}}{2} r\sin t, \quad y'_t = -\frac{\sqrt{2}}{2} r\sin t, \quad z'_t = r\cos t$$

$$(x'_t)^2 + (y'_t)^2 + (z'_t)^2 = \frac{1}{2}r^2\sin^2 t + \frac{1}{2}r^2\sin^2 t + r^2\cos^2 t = r^2\sin^2 t + r^2\cos^2 t$$
$$= r^2(\sin^2 t + \cos^2 t) = r^2$$

$$\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\psi'(t))^2} = \sqrt{r^2} = r$$

$$I = \int_c z\sqrt{x^2+y^2+2z^2}\,dS = \int_0^{\pi} r\sin t \sqrt{\underbrace{r^2\cos^2 t + 2r^2\sin^2 t}_{r^2(\cos^2 t + 2\sin^2 t)}}\; r\,dt =$$

$$= r^3 \int_0^{\pi} \sin t \sqrt{\cos^2 t + 2\underbrace{\sin^2 t}_{1-\cos^2 t}}\,dt = r^3 \int_0^{\pi} \sin t \sqrt{2-\cos^2 t}\,dt =$$

$$= \left| \begin{array}{l} \cos t = u \\ -\sin t\,dt = du \\ \sin t\,dt = -du \end{array} \quad t\Big|_0^{\pi} \Rightarrow u\Big|_1^{-1} \right| = r^3 \int_{-1}^{1} \sqrt{2-t^2}\,dt = r^3 \int_{-1}^{1} \frac{2-t^2}{\sqrt{2-t^2}}\,dt$$

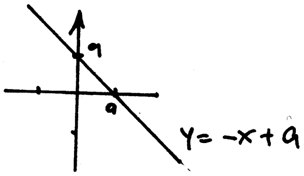
ZA VJEŽBU

$$\ldots = r^3 \cdot \frac{1}{2} t\sqrt{2-t^2}\Big|_{-1}^{1} + r^3 \int_{-1}^{1} \frac{dt}{2-t^2} = \ldots = \left(1+\frac{\pi}{2}\right) r^3$$

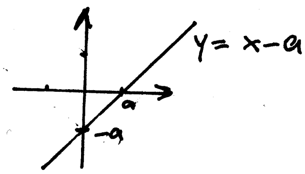
(#) Izračunati integral po krivoj c $\int\limits_c xy\,ds$ gdje je c kvadrat $|x|+|y|=a$, $a>0$.

Rj: Kako nacrtati kvadrat $|x|+|y|=a$?

1° $x>0$, $y>0$ $\quad x+y=a$



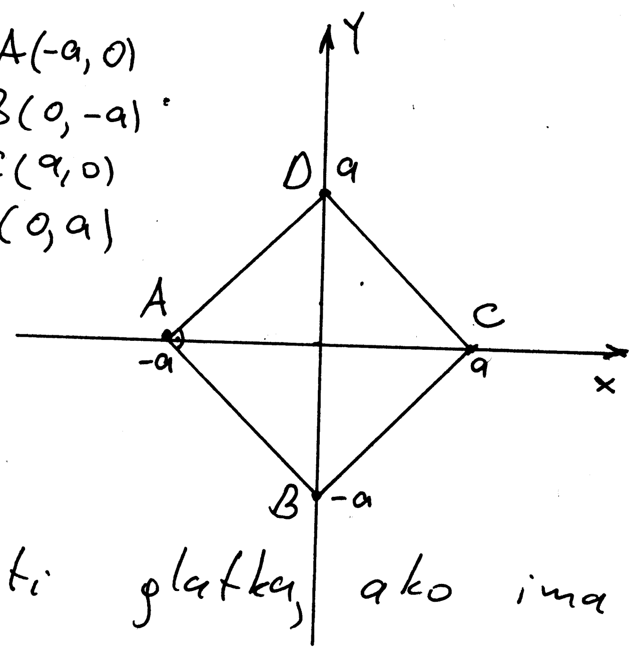
2° $x>0$, $y<0$ $\quad x-y=a$



3° $x<0$, $y>0$ $\quad -x+y=a$

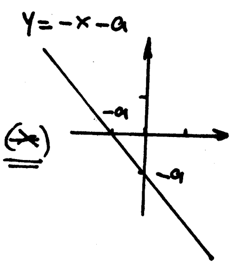
4° $x<0$, $y<0$ $\quad -x-y=a$

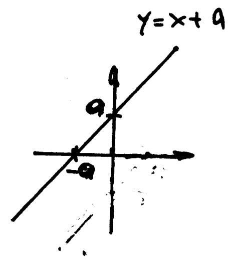
A(-a, 0)
B(0, -a)
C(a, 0)
D(0, a)



Kriva po kojoj se integrali mora biti glatka, ako ima čošak razbije se na dijelove.

$$\int\limits_c xy\,ds = \int\limits_{AB} xy\,ds + \int\limits_{BC} xy\,ds + \int\limits_{CD} xy\,ds + \int\limits_{DA} xy\,ds \overset{(*)}{=}$$





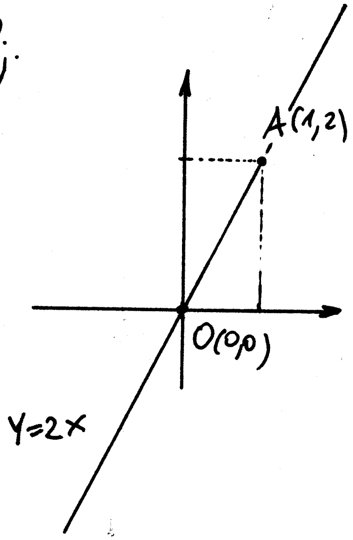
$$\boxed{\int\limits_c f(x,y)\,ds = \int_a^b f(x,\varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx}$$, gdje je c: $y=\varphi(x)$ kriva $x\in[a,b]$

$$\overset{(*)}{=} \int_{-a}^{0} x\cdot(-x-a)\sqrt{1+(-1)^2}\,dx + \int_0^a x(x-a)\sqrt{1+1^2}\,dx + \int_0^a x(-x+a)\sqrt{1+(-1)^2}\,dx$$

$$+ \int_{-a}^{0} x(x+a)\sqrt{1+1^2}\,dx = \sqrt{2}\left(\int_{-a}^{0}(-x^2-ax+x^2+ax)\,dx + \int_0^a (x^2-ax-x^2+ax)\,dx\right) = 0$$

Izračunati integral $\int_c \frac{ds}{\sqrt{x^2+y^2+4}}$ gdje je c duž koja spaja tačke $O(0,0)$ i tačku $A(1,2)$.

Rj.



c: $y=2x$. $y'=2$

$$\int_c f(x,y)\,ds = \int_a^b f(x,\varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx$$

gdje je $y=\varphi(x)$. kriva $x\in[a,b]$

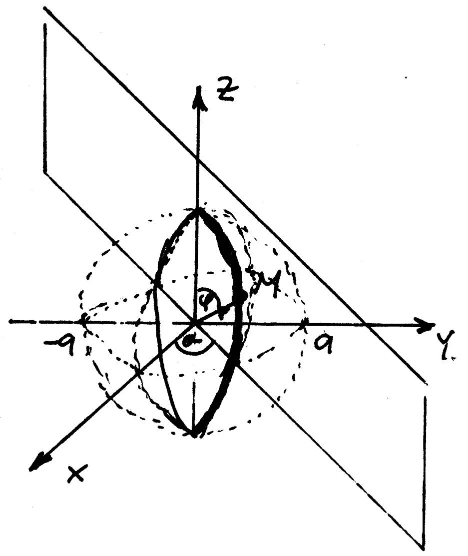
$$\int_c \frac{1}{\sqrt{x^2+y^2+4}}\,ds = \int_0^1 \frac{\sqrt{1+2^2}}{\sqrt{x^2+(2x)^2+4}}\,dx = \sqrt{5}\int_0^1 \frac{dx}{\sqrt{5x^2+4}} =$$

$$= \sqrt{5}\int_0^1 \frac{dx}{\sqrt{4(\frac{5}{4}x^2+1)}} = \frac{\sqrt{5}}{2}\int_0^1 \frac{d(\frac{\sqrt{5}}{2}x)}{\sqrt{(\frac{\sqrt{5}}{2}x)^2+1}}\cdot\frac{2}{\sqrt{5}} = \ln\left|\frac{\sqrt{5}x}{2}+\sqrt{\left(\frac{\sqrt{5}x}{2}\right)^2+1}\right|\Bigg|_0^1$$

$$= \ln\left|\frac{\sqrt{5}}{2}+\sqrt{\frac{5}{4}+1}\right| - \ln 1 = \ln\left|\frac{\sqrt{5}}{2}+\sqrt{\frac{9}{4}}\right| = \ln\frac{\sqrt{5}+3}{2}$$

(#) Izračunati $\int_c \sqrt{2y^2+z^2}\, ds$ gdje je c krug dobijen presjekom sfere $x^2+y^2+z^2=a^2$ i ravni $x=y$.

Rj.



Kako ćemo opisati datu sferu parametarski? (sferne koordinate)

$$x = r\sin\varphi\cos\theta \qquad r=a$$
$$y = r\sin\varphi\sin\theta \quad , \qquad 0\le\theta\le 2\pi$$
$$z = r\cos\varphi \qquad 0\le\varphi\le\pi$$

Kako da parametarski opišemo krug dobijen presjekom sfere i ravni?

Za pravu $x=y$ znamo da je ugao između ove prave i x-ose 45°. Prema tome $\theta=45^\circ$, $(r=a)$:

$$c:\begin{cases} x=\frac{\sqrt2}{2}a\sin\varphi \\ y=\frac{\sqrt2}{2}a\sin\varphi \\ z=a\cos\varphi \\ 0\le\varphi\le 2\pi \end{cases}$$

θ i r su fiksirani

$$2y^2+z^2 = 2\cdot\frac{2}{4}a^2\sin^2\varphi + a^2\cos^2\varphi = a^2$$

Ako je kriva c opisana sa $x=\mu(t)$, $y=\eta(t)$, $z=\xi(t)$, $\alpha<t<\beta$ onda je

$$\int_c f(x,y,z)\,ds = \int_\alpha^\beta f(\mu(t),\eta(t),\xi(t))\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\xi'(t))^2}\,dt$$

$$\frac{\partial x}{\partial\varphi} = \frac{\sqrt2}{2}a\cos\varphi$$

$$\frac{\partial y}{\partial\varphi} = \frac{\sqrt2}{2}a\cos\varphi$$

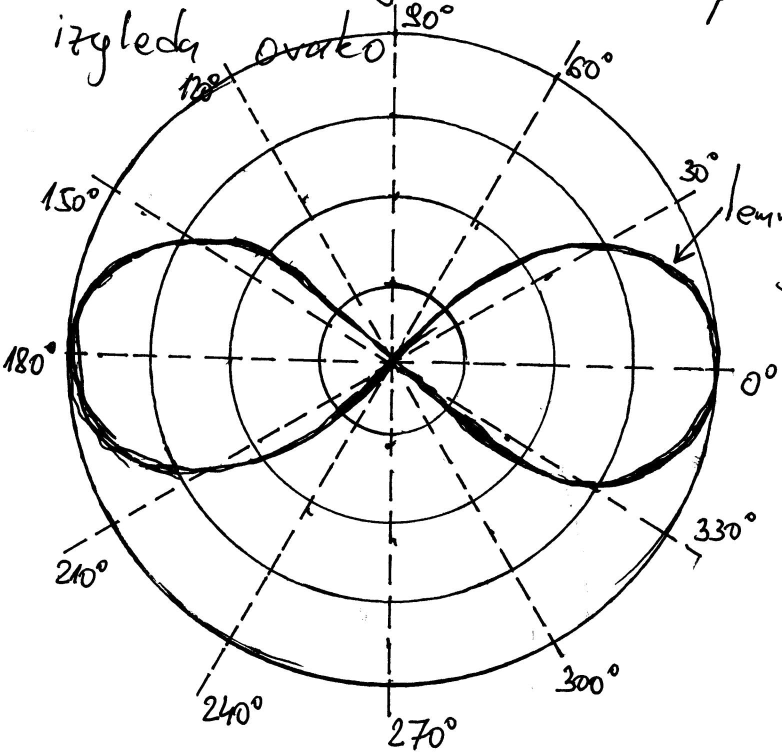
$$\frac{\partial z}{\partial\varphi} = -a\sin\varphi$$

$$\int_c \sqrt{2y^2+z^2}\,ds = \int_0^{2\pi}\sqrt{a^2}\cdot\sqrt{\tfrac{2}{4}a^2\cos^2\varphi+\tfrac{2}{4}a^2\cos^2\varphi+a^2\sin^2\varphi}\,dt$$

$$= \int_0^{2\pi} a\cdot\sqrt{a^2(\cos^2\varphi+\sin^2\varphi)}\,dt = a^2\int_0^{2\pi}dt = 2a^2\pi.$$

(#) Izračunati krivoliniski integral prve vrste $\int_c (x+y)\,dS$, ako je c desna latica lemniskate $\rho = a\sqrt{\cos 2\varphi}$.

Rj: Lemniskata $\rho = a\sqrt{\cos 2\varphi}$ u polarnom koordinatnom sistemu izgleda ovako



Data kriva je prikazana u polarnim koordinatama

$$c: \begin{cases} \rho = a\sqrt{\cos 2\varphi} \\ \varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \end{cases}$$

Prisjetimo se,

$$\int_c (x+y)\,dS = \int_{t_1}^{t_2} (\eta(t) + \mu(t))\sqrt{(\eta'(t))^2 + (\mu'(t))^2}\,dt$$

ako je c data u obliku

$$c: \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{cases}.$$

$$\int_{c'} (x+y)\,dS = \left| \begin{array}{l} \text{kao pomoć uvedimo polarne koordinate} \\ x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ \text{za } \rho \text{ ćemo uzeti} \\ \rho = a\sqrt{\cos 2\varphi} \end{array} \right.$$

Prema tome

desna latica lemniskate

$$c': \begin{cases} x = a\cos\varphi\sqrt{\cos 2\varphi} \\ y = a\sin\varphi\sqrt{\cos 2\varphi} \\ -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4} \end{cases}$$

zato što posmatramo desnu stranu lemniskate

$$x' = \left(a(-\sin\varphi)\sqrt{\cos 2\varphi} + a\cos\varphi \cdot \frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}} \cdot (-\sin 2\varphi)\cdot 2\right)d\varphi = a\left(-\sin\varphi\sqrt{\cos 2\varphi} - \cos\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right)d\varphi \overset{\text{adic. teor.}}{=} -a\frac{\sin 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$y' = \left(a\cos\varphi\sqrt{\cos 2\varphi} + a\sin\varphi \cdot \frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}} \cdot (-\sin 2\varphi)\cdot 2\right)d\varphi$$

$$= \left(a\cos\varphi\sqrt{\cos 2\varphi} - a\sin\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right)d\varphi \overset{\text{adiciona teor.}}{=} \dots = a\frac{\cos 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$x'^2 + y'^2 = a^2\frac{\sin^2 3\varphi}{\cos 2\varphi}\,d\varphi^2 + a^2\frac{\cos^2 3\varphi}{\cos 2\varphi}\,d\varphi^2 = a^2\frac{1}{\cos 2\varphi}\,d\varphi^2 \quad \Bigg| =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cancel{\sqrt{\cos 2\varphi}}\; a(\cos\varphi + \sin\varphi) \cdot a\frac{1}{\cancel{\sqrt{\cos 2\varphi}}}\,d\varphi = a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\varphi + \sin\varphi)\,d\varphi =$$

$$= a^2\Big(\underbrace{\sin\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}_{\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)} - \underbrace{\cos\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}_{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}\Big) = a^2\sqrt{2} \quad \text{traženo rješenje.}$$

Zadaci za vježbu

U zadacima 3770—3775 izračunati date krivolinijske integrale.

3770. $\int\limits_L \frac{ds}{x-y}$, pri čemu je L odsečak na pravoj $y=\frac{1}{2}x-2$, koji leži između tačaka $A(0, -2)$ i $B(4, 0)$.

3771. $\int\limits_L xy\,ds$, pri čemu je L kontura pravougaonika čija su temena $A(0, 0)$, $B(4, 0)$, $C(4, 2)$ i $D(0, 2)$.

3772. $\int\limits_L y\,ds$, pri čemu je L luk parabole $y^2=2px$, koji leži unutar parabole $x^2=2py$.

3773. $\int\limits_L (x^2+y^2)^n\,ds$, pri čemu je L krug $x=a\cos t$, $y=a\sin t$.

3774. $\int\limits_L xy\,ds$, pri čemu je L četvrtina elipse koja leži u prvom kvadrantu.

3775. $\int\limits_L \sqrt{2y}\,ds$, pri čemu je L prvi svod cikloide $x=a(t-\sin t)$,

$$y=a(1-\cos t).$$

3776. Napisati obrazac za izračunavanje integrala $\int\limits_L F(x, y)\,ds$ u polarnim koordinatama, ako je kriva L zadata jednačinom $\rho=\rho(\varphi)$ $(\varphi_1 \leqslant \varphi \leqslant \varphi_2)$.

3777*. Izračunati $\int\limits_L (x-y)\,ds$, po kružnoj liniji $x^2+y^2=ax$.

3778. Izračunati $\int\limits_L \sqrt{x^2-y^2}\,ds$ po krivoj $(x^2+y^2)^2=a^2(x^2-y^2)$ $(x \geqslant 0)$ (polovina lemniskate).

3779. Izračunati $\int\limits_L \operatorname{arctg}\frac{y}{x}\,ds$ po delu Arhimedove spirale $\rho=2\varphi$ koji leži unutar kruga poluprečnika R, čiji je centar u koordinatnom početku.

3780. Izračunati $\int\limits_L \frac{z^2\,ds}{x^2+y^2}$ po prvom zavoju zavojnice $x=a\cos t$, $y=a\sin t$, $z=at$.

3781. Izračunati $\int\limits_L xyz\,ds$ po delu kružne linije $x^2+y^2+z^2=R^2$, $x^2+y^2=\frac{R^2}{4}$, koji leži u prvom oktantu.

3782. Izračunati $\int\limits_L (2z-\sqrt{x^2+y^2})\,ds$ po prvom zavoju konusne zavojnice $x=t\cos t$, $y=t\sin t$, $z=t$.

3783. Izračunati $\int\limits_L (x+y)\,ds$ po delu kružne linije $x^2+y^2+z^2=R^2$, $y=x$, koji leži u prvom oktantu.

Rješenja

3770. $\sqrt{5}\ln 2$. **3771.** 24.

3772. $\frac{p^2}{3}(5\sqrt{5}-1)$. **3773.** $2\pi a^{2n+1}$.

3774. $\frac{ab(a^2+ab+b^2)}{3(a+b)}$. **3775.** $4\pi a\sqrt{a}$.

3776. $\int\limits_{\varphi_1}^{\varphi_2} F(\rho\cos\varphi, \rho\sin\varphi)\sqrt{\rho^2+\rho'^2}\,d\varphi$.

3777*. $\frac{\pi a^2}{2}$. Preći na polarne koordinate.

3778. $\frac{2a^3\sqrt{2}}{3}$. 3779. $\frac{1}{12}[(R^2+4)^{\frac{3}{2}}-8]$.

3780. $\frac{8a\pi^2\sqrt{2}}{3}$. 3781. $\frac{R^4\sqrt{3}}{32}$

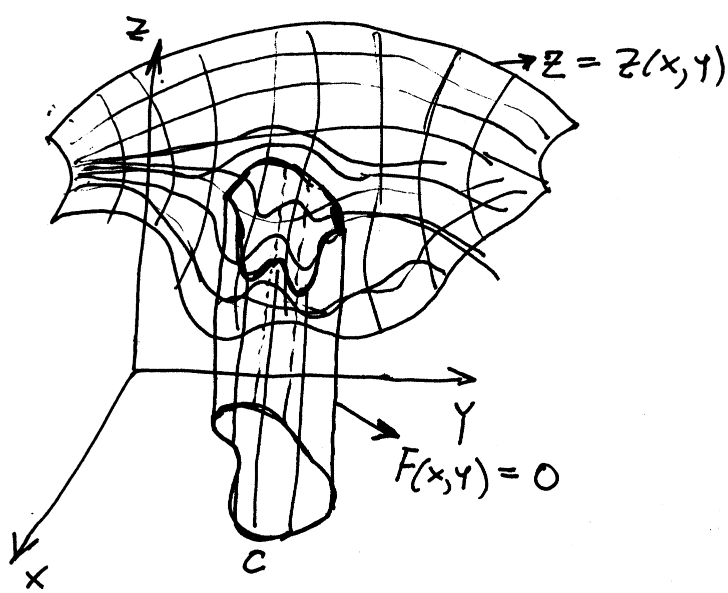
3782. $\frac{2\sqrt{2}}{3}[(1+2\pi^2)^{\frac{3}{2}}-1]$. 3783. $R^2\sqrt{2}$.

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Računanje površine cilindrične površi

Ako je S dio cilindrične površine $F(x,y)=0$ između xOy ravni i neke površine $z=z(x,y)$ tada se površina $P(S)$ površi S računa po formuli

$$P(S)=\int_C z(x,y)\,dS \quad \text{gdje je} \quad C:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}.$$



$P(S)$ - površina dijela cilindrične površi

Izračunati površinu eliptičkog valjka $9x^2+5y^2=45$ kojii se nalazi između površi $z=0$ i $z=y$.

Rj.

$$P(s)=\int_C z(x,y)\,dS \quad \text{gdje je} \quad c:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}.$$

Skicirajmo valjak $9x^2+5y^2=45$. $/:45$ u xOy ravni on izgleda

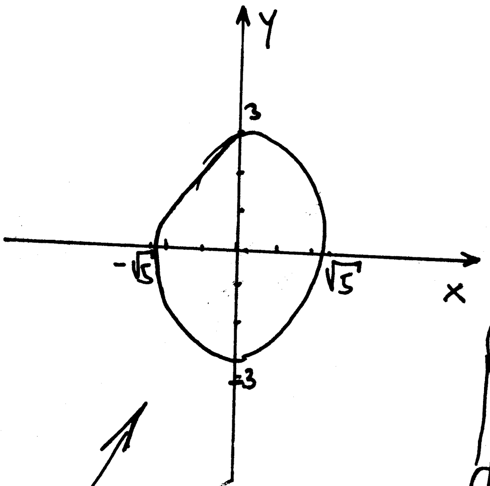
$z=0$ je xOy ravan

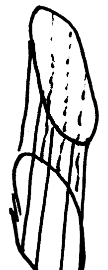
$z=y$ u yOz ravni

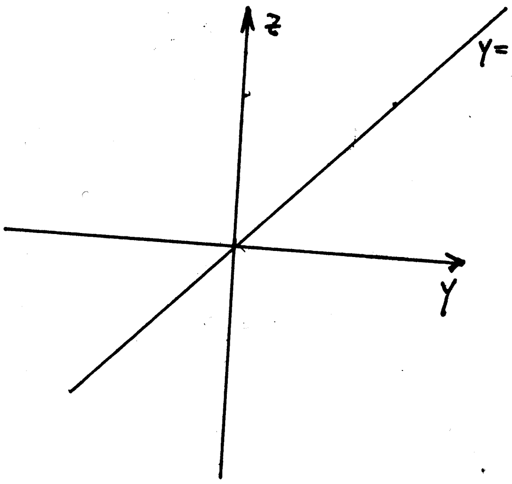
$$\frac{x^2}{5}+\frac{y^2}{9}=1$$

elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\sqrt{5}\approx 2{,}23$







$z(x,y)=y$

$$c:\begin{cases} 9x^2+5y^2=45 \\ z=0 \end{cases}$$

c je elipsa

$z=y$ je ravan koja sadrži x-osu a u yOz ravni sadrži $y=z$ pravu

Svedimo elipsu $\frac{x^2}{5}+\frac{y^2}{9}=1$ na parametarski oblik $\begin{matrix} x=a\cos t \\ y=b\sin t \end{matrix}$

U našem slučaju $\begin{matrix} x=\sqrt{5}\cos t \\ y=3\sin t \\ 0\le t\le 2\pi \end{matrix}$

$$c:\begin{cases} x=\mu(t) \\ y=\eta(t) \\ t_1\le t\le t_2 \end{cases} \quad \int_C f(x,y)\,ds=\int_{t_1}^{t_2} f(\mu(t),\eta(t))\underbrace{\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt}_{ds}$$

površina figure koja se nalazi u I i II oktantu

$$dS=\sqrt{5\sin^2 t+9\cos^2 t}\,dt$$

Kako se ravni $z=0$ i $z=y$ sijeku u x-osi to će parametar t uzimati vrijednosti od 0 do π

$$P(s)=\int_C y\,dS=\int_0^{\pi} 3\sin t\sqrt{5\sin^2 t+9\cos^2 t}\,dt=3\int_0^{\pi}\sin t\sqrt{5(1-\cos^2 t)+9\cos^2 t}\,dt=$$

$$=3\int_0^{\pi}\sin t\sqrt{5+4\cos^2 t}\,dt=\left|\begin{matrix} 2\cos t=u \\ -2\sin t\,dt=du \\ \sin t\,dt=-\frac{1}{2}u \end{matrix}\quad \begin{matrix} t=0 \Rightarrow u=2 \\ t=\pi \Rightarrow u=-2 \end{matrix}\right|=3\int_2^{-2}\left(-\frac{1}{2}\right)\underbrace{\sqrt{5+u^2}}_{\text{parna f-ja}}\,du=$$

$$=3\cdot\frac{1}{2}\cdot 2\int_0^2\sqrt{5+u^2}\,du \overset{\text{umjesto } u \text{ stavimo } x}{=} 3\int_0^2\frac{5+x^2}{\sqrt{5+x^2}}\,dx=3\int_0^2\frac{5}{\sqrt{5+x^2}}\,dx+3\int_0^2\frac{x^2}{\sqrt{5+x^2}}=\left|\begin{matrix} u=x & dv=\frac{x}{\sqrt{5+x^2}} \\ \text{ZAVRŠITI} & \\ \text{SAMI} \ldots & \end{matrix}\right|=\frac{15\sqrt{45}}{4}+9\ldots$$

(#) Izračunati površinu dijela valjka $x^2+y^2=1$ koji se nalazi između površi $z=0$ i $z=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$

Rj: $P(S)=\int_c z(x,y)\,dS$ gdje je $c:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}$

U ovom slučaju je $z(x,y)=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$

$c:\begin{cases} x^2+y^2=1 \\ z=0 \end{cases}$ tj. $c: x^2+y^2=1$

Parametrizirajmo kružnicu: $x=r\cos\varphi$, $y=r\sin\varphi$, $0\le\varphi\le 2\pi$

U našem slučaju: $x=\cos\varphi$, $y=\sin\varphi$, $0\le\varphi\le 2\pi$

$c:\begin{cases} x=\mu(t) \\ y=\eta(t) \\ t_1\le t\le t_2 \end{cases}$

$$\int_c f(x,y)\,dS=\int_{t_1}^{t_2} f(\mu(t),\eta(t))\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt$$

$(\cos\varphi)'=-\sin\varphi$

$(\sin\varphi)'=\cos\varphi$

$$dS=\sqrt{\sin^2\varphi+\cos^2\varphi}\,d\varphi=d\varphi$$

$\sqrt{x^2+y^2}=1$

$\sqrt{1-x^2}=\cos\varphi$ | $\sqrt{1-y^2}=\sin\varphi$

Definiciono područje f-je $z=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$ je $\{(x,y)\,|\,-1\le x\le 1 \text{ i } -1\le y\le 1\}$

zbog simetričnosti četiri dijela

$$P(S)=\int_c\left(\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}\right)dS=4\int_0^{\frac{\pi}{2}}(1+\sin\varphi+\cos\varphi)\,d\varphi=$$

$$=4\left[\varphi\Big|_0^{\frac{\pi}{2}}-\cos\varphi\Big|_0^{\frac{\pi}{2}}+\sin\varphi\Big|_0^{\frac{\pi}{2}}\right]=4\left(\frac{\pi}{2}+1+1\right)=2\pi+8$$

(#)^{v} Izračunati površinu cilindra $x^2+y^2=R^2$ između ravni $z=0$ i površi $z=R+\frac{x^2}{R}$.

Zadaci za vježbu

U zadacima 3792 — 3797 izračunati površine datih cilindričnih omotača, koji leže između ravni Oxy i navedenih površina.

3792. $x^2+y^2=R^2, \quad z=R+\dfrac{x^2}{R}.$

3793. $y^2=2px, \quad z=\sqrt{2px-4x^2}.$

3794. $y^2=\dfrac{4}{9}(x-1)^3, \quad z=2-\sqrt{x}.$

3795. $x^2+y^2=R^2, \quad 2Rz=xy.$

3796. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, \quad z=kx$ i $z=0$ $(z\geqslant 0)$ („cilindrična potkovica")

3797. $y=\sqrt{2px}, \quad z=y$ i $x=\dfrac{8}{9}p.$

3798. Izračunati površinu onog dela kružnog cilindra koji iz njega iseca drugi isti takav cilindar, ako im se ose seku pod pravim uglom a poluprečnici su im R (uporedi sa rešenjem zadatka 3642).

3799. Naći površinu onog dela cilindra $x^2+y^2=Rx$, koji leži unutar sfere $x^2+y^2+z^2=R^2$.

Rješenja

3792. $3\pi R^2$. **3793.** $\dfrac{\pi p^2}{4}$. **3794.** $\dfrac{11}{3}$. **3795.** R^2.

3796. $ka\left(a+\dfrac{b^2}{2c}\ln\dfrac{a+c}{a-c}\right)$, gde je $c=\sqrt{a^2-b^2}$. Za $a=b$ $S=2ka^2$.

3797. $\dfrac{98}{81}p^2$. **3798.** $8R^2$. **3799.** $4R^2$.